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## **$c=1$ discrete states correlators via $W_{1+\infty}$ constraints**

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### **Abstract**

The discrete states of  $c = 1$  string theory at the self-dual radius are associated with modes of  $W_{1+\infty}$  currents and their genus zero correlators are computed. An analogy to a recent suggestion based on the integrable structure of the theory is found. An iterative method for deriving the dependence of the currents on the full space of couplings is presented and applied. The dilaton equation of the theory is derived.

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The discrete states of  $c = 1$  string theory have been originally found in the Liouville formulation [1–5] and interpreted as  $2d$  remnants of transverse massive string excitations [6, 7]. The computation of their correlators in the presence of the cosmological constant in this framework is rather non-trivial and has not been carried out successfully yet.

The interpretation of the discrete states as gravitational descendants in the topological description of  $2d$  string theory at the self-dual radius [9, 10] opened a way to the computation of the correlators by other means. In [11] the correlators have been computed by using topological recursion relations derived via analytical continuation of those of the minimal models.

Discrete states have been introduced in the matrix model approach by associating them with powers of the eigenvalue matrix [12, 13]. Their correlators have been calculated in this formulation with results that do not coincide with those of [11].

In this letter we take another route for defining the discrete states. In analogy with  $(1, q)$  minimal topological models in which gravitational descendants are associated with modes of the  $W_q$  currents [14–16], we associate the discrete states with modes of the  $W_{1+\infty}$  constraints algebra of  $c = 1$  string theory [17] and compute their correlators. We find analogy to the suggestion made in [18] based on the integrable structure of the theory. The results that we get for the discrete states correlators do not agree neither with [11] nor with [12, 13], and the implications of this will be discussed at the end of the paper.

The  $W$  constraints on the partition function of minimal topological matter coupled to topological gravity read [14–16].

$$\mu^{-2} Z^{-1} \partial_{k,\alpha} Z = \langle \langle \sigma_{k,\alpha} \rangle \rangle = Z^{-1} W_{k-\alpha}^{(\alpha+1)} Z . \quad (1)$$

$\sigma_{k,\alpha}$  is the  $k^{th}$  gravitational descendant of the primary field  $O_\alpha$  and  $W_m^{(\alpha+1)}$  is the  $m^{th}$  mode of the spin  $\alpha + 1$  current. The partition function is:

$$\log Z(t) = \mu^2 \langle \exp(\sum_{k,\alpha} t_{k,\alpha} \sigma_{k,\alpha}) \rangle , \quad (2)$$

and

$$\langle \cdots \rangle \equiv \sum_{g \geq 0} \frac{1}{\mu^{2g}} \langle \cdots \rangle_g , \quad (3)$$

with  $g$  being the genus of the Riemann surface.

The  $W_{1+\infty}$  Ward identities for the tachyon correlators in  $2d$  string theory read [17]

$$\langle \langle \mathcal{T}_n \rangle \rangle = Z^{-1} \bar{W}_{-n}^{(n+1)} Z , \quad (4)$$

where  $\bar{W}^{(n+1)}$  is the spin  $n + 1$  current of a  $W_{1+\infty}$  algebra and

$$\langle\langle \mathcal{T}_n \rangle\rangle = \langle \mathcal{T}_n \exp(\sum_{k=-\infty}^{\infty} t_k \mathcal{T}_k) \rangle . \quad (5)$$

By the tachyon  $\mathcal{T}_n$  we mean the Seiberg state  $\mathcal{T}_n^+$  with positive momentum  $n$  [19]. Analogous  $W_{1+\infty}$  Ward identities exist for a negative momentum tachyon  $\mathcal{T}_{-n}^+$

$$\langle\langle \mathcal{T}_{-n} \rangle\rangle = Z^{-1} W_{-n}^{(n+1)} Z , \quad (6)$$

where  $W^{(n+1)}$  is the spin  $n + 1$  current of a similar  $W_{1+\infty}$  algebra.

Comparing the Ward identities (1) for the primary operators  $\sigma_{0,\alpha}$  and (4) we are led to identify  $\mathcal{T}_n$  as primaries in the topological description of  $2d$  string theory. This identification has been made in [9] using integrable and topological reasoning. In analogy with (1), we may try to generalize (4) by using the other modes of the  $W_{1+\infty}$  currents, namely

$$\langle\langle \sigma_k(\mathcal{T}_n) \rangle\rangle = Z^{-1} \bar{W}_{k-n}^{(n+1)} Z . \quad (7)$$

Upon identifying the discrete states as gravitational descendants in the topological formulation of the theory [9, 10]

$$\mathcal{Y}_{J,m} = \sigma_k(\mathcal{T}_n) , \quad (8)$$

with  $k = J - m, n = J + m$ , we have

$$\langle\langle \mathcal{Y}_{J,m} \rangle\rangle = Z^{-1} \bar{W}_{-2m}^{(J+m+1)} Z , \quad (9)$$

with  $J$  taking half integer values and  $-J \leq m \leq J$ .

The currents in (1) depend on the times associated with both primaries and descendants, i.e. they are defined on the full phase space. Thus, we will assume in the paper that equation (9) holds on the full phase space too. From the  $W_{1+\infty}$  Ward identities we only know the dependence of the currents in (9) on the times associated with the tachyons. As we will show in the sequel we can in fact find the dependence of the currents in (9) on the times associated with the discrete states, thus enlarging the  $W_{1+\infty}$  Ward identities to the full phase space

$$\langle\langle \mathcal{Y}_{J,m} \rangle\rangle = Z^{-1} \bar{\mathcal{W}}_{-2m}^{(J+m+1)} Z , \quad (10)$$

where on the space of only tachyon times  $t_n$ ,  $\bar{\mathcal{W}}_{-2m}^{(J+m+1)}(t_n) \equiv \bar{W}_{-2m}^{(J+m+1)}(t_n)$ .

In the following we will consider the genus zero case. Equation (9) on the space of tachyon times reads [17]

$$\langle\langle \mathcal{Y}_{J,m} \rangle\rangle_0 = \frac{1}{J+m+1} \oint x^{J-m} \bar{W}^{J+m+1} , \quad (11)$$

where

$$\bar{W} = \frac{1}{x} [1 + \sum_{k>0} k t_{-k} x^k + \sum_{k>0} x^{-k} \langle \langle \mathcal{T}_{-k} \rangle \rangle_0] . \quad (12)$$

A priori, equation (11) should be used only for positive momentum discrete states  $\mathcal{Y}_{J,m}, m > 0$ , while for negative momenta its parity transformed version

$$\langle \langle \mathcal{Y}_{J,-m} \rangle \rangle_0 = \frac{1}{J+m+1} \oint x^{J-m} W^{J+m+1} , \quad (13)$$

where

$$W = \frac{1}{x} [1 + \sum_{k>0} k t_k x^k + \sum_{k>0} x^{-k} \langle \langle \mathcal{T}_k \rangle \rangle_0] , \quad (14)$$

should be used. The  $c = 1$  string equation [18, 20], which can be written in the form [9]

$$x = \bar{W}(W(x)) , \quad (15)$$

leads to the identity<sup>†</sup>

$$\frac{1}{J+m+1} \oint x^{J-m} \bar{W}^{J+m+1} = \frac{1}{J-m+1} \oint x^{J+m} W^{J-m+1} , \quad (16)$$

and therefore we can use (9) for negative values of  $m$  as well.

It is interesting to notice that equation (11) coincides with the suggestion of [18] for discrete states correlators. The latter was based on studying the symmetries of the Toda lattice hierarchy in the dispersionless limit.

Equation (10) provides a complete and self-contained definition of all the tachyon and discrete states correlators. Equation (11) is valid on the full phase space, with  $\bar{W}(t_n)$  being replaced by  $\bar{\mathcal{W}}(t_{J,m})$  which we have to construct. Note that we implicitly assume that the string equation (15) holds on the full phase space too

$$x = \bar{\mathcal{W}}(\mathcal{W}(x)) . \quad (17)$$

Equation (11) for the negative momentum tachyon  $\mathcal{T}_{-n=-2J} \equiv \mathcal{Y}_{J,-J}$  on the full phase space reads:

$$\langle \langle \mathcal{T}_{-n} \rangle \rangle_0 = \oint x^n \bar{\mathcal{W}} . \quad (18)$$

This implies that

$$\bar{\mathcal{W}} = \frac{1}{x} [1 + \sum_{k>0} x^k U_k(t_{J,m}) + \sum_{k>0} x^{-k} \langle \langle \mathcal{T}_{-k} \rangle \rangle_0] , \quad (19)$$

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<sup>†</sup>The identity (16) is a special case of a more general one :  $\oint f(x)G(W) = \oint g(x)F(\bar{W})$ , where  $f(x), g(x)$  are any two functions and  $F'(x) = f(x), G'(x) = g(x)$ .

where  $\langle\langle \cdot \cdot \rangle\rangle$  is defined on the full phase space, and  $U_k$  are unknown functions. It is tempting to claim, using (11) with  $J \rightarrow -J$ , that the functions  $U_k$  in (19) are given by

$$U_k(t_{J,m}) = \langle\langle \mathcal{Y}_{-J,J} \rangle\rangle_0 \equiv \langle\langle \mathcal{T}_{k=2J}^- \rangle\rangle_0 , \quad (20)$$

where  $\mathcal{T}_k^-$  is an anti-Seiberg state. This is incorrect, since due to (12),  $U_k = kt_{-k}$  on the space of tachyon times. This together with (20) implies the vanishing of the correlators  $\langle\mathcal{T}_n^- \prod_{i=1}^s \mathcal{T}_{n_i}^+ \rangle = 0$  for  $s > 2$  which is in contradiction with calculations of bulk correlators in the Liouville formulation [6, 7, 21].  $U_k$  can be determined perturbatively in  $t_{J,m}$  as we will show now.

Consider first correlators with insertion of one discrete state and a few tachyons by using

$$\langle\mathcal{Y}_{J,m} \prod_{i=1}^l \mathcal{T}_{n_i} \rangle_g = \partial_{n_1} \cdots \partial_{n_l} \langle\langle \mathcal{Y}_{J,m} \rangle\rangle_g(t=0) . \quad (21)$$

Using (11),(12) and (13),(14) we calculate the first few genus zero correlators:

$$\begin{aligned} \langle\mathcal{Y}_{J,m} \rangle_0 &= \frac{1}{J+1} \delta_{m,0} \\ \langle\mathcal{Y}_{J,m} \mathcal{T}_n \rangle_0 &= 2|m| \delta_{n+2m,0} \\ \langle\mathcal{Y}_{J,m} \mathcal{T}_{n_1} \mathcal{T}_{n_2} \rangle_0 &= (J+|m|) |n_1 n_2| \delta_{n_1+n_2+2m,0} \\ \langle\mathcal{Y}_{J,m} \prod_{i=1}^3 \mathcal{T}_{n_i} \rangle_0 &= (J+|m|) \prod_{i=1}^3 |n_i| (J-|m|+2m-1 - \sum_{j=1}^3 (2m+n_j) \\ &\quad \theta(-2m-n_j)) \delta_{n_1+n_2+n_3+2m,0} . \end{aligned} \quad (22)$$

The first three correlators in (22) are explicitly parity invariant, ( $m \leftrightarrow -m, n_i \leftrightarrow -n_i$ ), while the four point function is parity invariant as a consequence of the identity  $2m - \sum_{j=1}^3 (2m+n_j) \theta(-2m-n_j) = \frac{1}{2} \sum_{i=1}^3 |2m+n_i|$ . The dependence of the  $W_{1+\infty}$  constraints on the times associated with the discrete states can be found iteratively as follows: Consider

$$\langle\mathcal{Y}_{J,-m} \mathcal{T}_n \rangle_0 = 2m \delta_{n,2m} = \partial_{J,-m} \langle\langle \mathcal{T}_n \rangle\rangle_0(t=0) , \quad (23)$$

where  $\partial_{J,m} \equiv \frac{\partial}{\partial t_{J,m}}$ . It implies that we have to modify  $\bar{W}$  by adding to it the term

$$\sum_{J,0 < m \leq J} 2mt_{J,-m} x^{2m-1} . \quad (24)$$

For  $J = m$  it coincides with the corresponding term for the tachyons. Using

$$\langle\mathcal{Y}_{J,m} \mathcal{T}_{-n} \rangle_0 = 2m \delta_{n,2m} = \partial_{J,m} \langle\langle \mathcal{T}_{-n} \rangle\rangle_0(t=0) , \quad (25)$$

we see that we have to add a similar term of the form

$$\sum_{J; 0 < m \leq J} 2mt_{J,m}x^{-2m-1} . \quad (26)$$

Thus,

$$\bar{\mathcal{W}} = \frac{1}{x} [1 + \sum_{J; -J \leq m \leq J} 2|m|t_{J,m}x^{-2m} + O(t^2)] . \quad (27)$$

This can be used in order to compute the two-point function of discrete states

$$\langle \mathcal{Y}_{J_1,m} \mathcal{Y}_{J_2,-m} \rangle_0 = \partial_{J_1,m} \langle \mathcal{Y}_{J_2,-m} \rangle_0(t=0) = 2|m| . \quad (28)$$

Consider now the correlator  $\langle \mathcal{Y}_{J,m} \mathcal{T}_{n_1} \mathcal{T}_{n_2} \rangle_0$ . We can use it in order to get the next correction to  $\bar{\mathcal{W}}$  that takes the form

$$\sum_{J;m,n} C_{J,m,n} t_{J,m} t_n x^{-2m-n-1} , \quad (29)$$

where

$$C_{J,m,n} = |n| |n + 2m| [(J + |m|)\theta(2m + n) + (J - |m|)\theta(-2m - n)] . \quad (30)$$

Using (30) we can calculate the three point function

$$\langle \mathcal{Y}_{J_1,m_1} \mathcal{Y}_{J_2,m_2} \mathcal{T}_n \rangle_0 = 4(J_1|m_2| + J_2|m_1|)|m_1 + m_2| \delta_{2m_1+2m_2+n,0} . \quad (31)$$

From (31) we deduce the full order  $(t^2)$  term in  $\bar{\mathcal{W}}$ , it reads

$$\sum_{J_1,J_2;m_1,m_2} C_{J_1,m_1,J_2,m_2} t_{J_1,m_1} t_{J_2,m_2} x^{-2m_1-2m_2-1} , \quad (32)$$

with

$$C_{J_1,m_1,J_2,m_2} = 2(J_1|m_2| + J_2|m_1|)|m_1 + m_2| + 4|m_1 m_2|(m_1 + m_2)\theta(-m_1 - m_2) . \quad (33)$$

The three point function of discrete states is thus calculated to be

$$\langle \mathcal{Y}_{J_1,m_1} \mathcal{Y}_{J_2,m_2} \mathcal{Y}_{J_3,m_3} \rangle_0 = 4(J_1|m_2 m_3| + J_2|m_1 m_3| + J_3|m_1 m_2| - |m_1 m_2 m_3|) , \quad (34)$$

where momentum conservation implies  $m_1 + m_2 + m_3 = 0$ . When one of the discrete states is a tachyon, equation (34) reduces to (31), while if two of them are tachyons it reduces to (22). Note that formally

$$C_{J_1,m_1,J_2,m_2} = \frac{1}{2} \langle \mathcal{Y}_{J_1,m_1} \mathcal{Y}_{J_2,m_2} \mathcal{Y}_{m_1+m_2,-m_1-m_2} \rangle_0 , \quad (35)$$

as indeed follows from the definition (11). Such relations hold for higher point functions too, and are useful for extracting the structure of  $\bar{W}$  from the discrete states correlators.

The same procedure leads to the four point function of discrete states

$$\begin{aligned} \langle \mathcal{Y}_{J_1, m_1} \mathcal{Y}_{J_2, m_2} \mathcal{Y}_{J_3, m_3} \mathcal{Y}_{J_4, m_4} \rangle_0 &= 8|m_1 m_2 m_3| [J_4^2 - m_4^2 - J_4(2 \max\{|m_i|\} + 1)] + perm \\ &+ 8J_1 J_2 |m_3 m_4| (2 \max\{|m_i|\} - |m_1 + m_2|) + perm \\ &+ 16|m_1 m_2 m_3 m_4| (2 \max\{|m_i|\} + 1) \end{aligned} \quad (36)$$

When three of the operators are tachyons the correlation function (36) reduces to the equation (22). The full order ( $t^3$ ) term in  $\bar{W}$  reads

$$\sum_{J_1, J_2, J_3; m_1, m_2, m_3} C_{J_1, m_1, J_2, m_2, J_3, m_3} t_{J_1, m_1} t_{J_2, m_2} t_{J_3, m_3} x^{-2m_1 - 2m_2 - 2m_3 - 1}, \quad (37)$$

with

$$C_{J_1, m_1, J_2, m_2, J_3, m_3} = \frac{1}{6} \langle \mathcal{Y}_{J_1, m_1} \mathcal{Y}_{J_2, m_2} \mathcal{Y}_{J_3, m_3} \mathcal{Y}_{m_1 + m_2 + m_3, -m_1 - m_2 - m_3} \rangle_0. \quad (38)$$

The iterative procedure described above can be used to find the complete dependence of the  $W_{1+\infty}$  currents on the full phase space.

The description of  $c = 1$  string theory at the self-dual radius as a topological field theory should include the puncture and dilaton equations. It has been observed that the momentum one tachyon corresponds to the puncture operator [22, 9], thus the  $\mathcal{T}_1$  Ward identity is the puncture equation. We will now derive the dilaton equation. The dilaton is the first descendant of the puncture. Thus, consider

$$\langle \langle \mathcal{Y}_{1,0} \rangle \rangle = Z^{-1} \bar{W}_0^{(2)} Z, \quad (39)$$

where  $\bar{W}_0^{(2)}$  is the zero mode of the Virasoro current. Equation (39) reads to all genera [17]

$$\begin{aligned} \langle \langle \mathcal{Y}_{1,0} \rangle \rangle &= \frac{1}{2} \oint \frac{1}{x} [1 + \sum_{k>0} k t_{-k} x^k + \sum_{k>0} x^{-k} \langle \langle \mathcal{T}_{-k} \rangle \rangle]^2 = \\ &= \frac{1}{2} + \sum_{g; k>0} \frac{1}{\mu^{2g}} k t_{-k} \langle \langle \mathcal{T}_{-k} \rangle \rangle_g. \end{aligned} \quad (40)$$

It yields

$$\langle \mathcal{Y}_{1,0} \prod_{i=1}^m \mathcal{T}_{n_i} \rangle_g = \partial_{n_1} \cdots \partial_{n_m} \langle \langle \mathcal{Y}_{1,0} \rangle \rangle_g(t=0) = \left( \frac{1}{2} \sum_{i=1}^m |n_i| \right) \langle \prod_{j=1}^m \mathcal{T}_{n_j} \rangle_g. \quad (41)$$

Using the  $\mathcal{T}_0$  Ward identity<sup>‡</sup>

$$\langle \mathcal{T}_0 \prod_{i=1}^m \mathcal{T}_{n_i} \rangle_g = \left( \frac{1}{2} \sum_{i=1}^m |n_i| + 2 - 2g - m \right) \langle \prod_{j=1}^m \mathcal{T}_{n_j} \rangle_g , \quad (42)$$

with (41), we see that the operator  $\mathcal{D} \equiv \mathcal{T}_0 - \mathcal{Y}_{1,0}$  satisfies

$$\langle \mathcal{D} \prod_{i=1}^m \mathcal{T}_{n_i} \rangle_g = (2 - 2g - m) \langle \prod_{j=1}^m \mathcal{T}_{n_j} \rangle_g . \quad (43)$$

Equation (43) is the dilaton equation and  $\mathcal{D}$  is the dilaton operator which measures the Euler characteristic of the Riemann surface with  $m$  punctures, that is  $2 - 2g - m$ .

There are several questions that arise as a consequence of this work. We presented an iterative method for computing the constraints currents on the full phase space and carried the computation to a certain order in  $t_{J,m}$ . It would be interesting to find the complete constraints algebra on the full phase space and for arbitrary genus. This algebra should have a topological interpretation along the lines of [11] or [24], and an underlying integrable hierarchy generalizing the Toda lattice, that should be discovered. For that the integrable viewpoint of [18, 23] is likely to be helpful. Furthermore, one expects these results to be intimately related to intersection theory on the moduli space of Riemann surfaces.

The results for discrete states correlators as calculated in this letter do not coincide with those calculated in [11, 13]. This implies that there is more than one way to perturb the tachyon theory, by introducing extra operators. Thus, the main, unanswered yet, question is which of the various theories is equivalent to the Liouville  $c = 1$  string theory.

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<sup>‡</sup>This identity can be derived, for instance in the continuum, by shifting the constant mode of the Liouville field.



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